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A-Statistical extension of the Korovkin type approximation theorem

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Abstract. In this paper, using the concept of A-statistical convergence which is a regular (non-matrix) summability method, we obtain a general Korovkin type approximation theorem which concerns the problem of approximating a function f by means of a sequence $\{L_n f\}$ of positive linear operators.

Keywords. A-statistical convergence; positive linear operator; Korovkin theorem.

1. Introduction

Let $A = (a_{jn})$ be an infinite summability matrix. For a given sequence $x := (x_n)$, the *A-transform of x*, denoted by $Ax := ((Ax)_j)$, is given by $(Ax)_j = \sum_{n=1}^{\infty} a_{jn}x_n$ provided the series converges for each $j \in \mathbb{N}$, the set of all natural numbers. We say that A is *regular* if $\lim Ax = L$ whenever $\lim x = L$ [4]. Assume that A is a non-negative regular summability matrix. Then $x = (x_n)$ is said to be *A-statistically convergent to L* if, for every $\varepsilon > 0$, $\lim_j \sum_{n:|x_n - L| \geq \varepsilon} a_{jn} = 0$, which is denoted by $st_A - \lim x = L$ [9] (see also [13,16]). We note that by taking $A = C_1$, the Cesàro matrix, A-statistical convergence reduces to the concept of *statistical convergence* (see [8,10,17] for details). If A is the identity matrix, then A-statistical convergence coincides with the ordinary convergence. It is not hard to see that every convergent sequence is A-statistically convergent. However, Kolk [13] showed that A-statistical convergence is stronger than convergence when $A = (a_{jn})$ is a regular summability matrix such that $\lim_j \max_n |a_{jn}| = 0$. It should be noted that A-statistical convergence may also be given in normed spaces [14].

Approximation theory has important applications in the theory of polynomial approximation, in various areas of functional analysis [1,3,5,12,15]. The study of the Korovkin type approximation theory is a well-established area of research, which deals with the problem of approximating a function f by means of a sequence $\{L_n f\}$ of positive linear operators. Statistical convergence, which was introduced nearly fifty years ago, has only recently become an area of active research. Especially it has made an appearance in approximation theory (see, for instance, [7,11]). The aim of the present paper is to investigate their use in approximation theory settings.

Throughout this paper $I := [0, \infty)$. As usual, let $C(I) := \{f\colon f$ is a real-valued continuous functions on $I\}$, and $C_B(I) := \{f \in C(I)\colon f$ is bounded on $I\}$. Consider the space H_w of

all real-valued functions f defined on I and satisfying

$$|f(x)-f(y)| \leq w\left(f;\left|\frac{x}{1+x}-\frac{y}{1+y}\right|\right),$$

where w is the modulus of continuity given by, for any $\delta > 0$,

$$w(f;\delta) := \sup_{\substack{x,y\in I\\ |x-y|\leq\delta}} |f(x)-f(y)|.$$

Assume that L is a linear operator mapping H_w into $C_B(I)$. As usual, we say that L is a *positive* linear operator provided that $f \geq 0$ implies $Lf \geq 0$, $f \in H_w$. Also, we denote the value of Lf at a point $x \in I$ by $L(f(u);x)$ or simply $L(f;x)$.

We now recall that Çakar and Gadjiev [6] obtained the following result.

Theorem A. *Let $\{L_n\}$ be a sequence of positive linear operators from H_w into $C_B(I)$. Then, the uniform convergence of the sequence $\{L_n f\}$ to f on I holds for any function $f \in H_w$ if the sequences $\{L_n\varphi_i\}$ converge uniformly to φ_i, where $\varphi_i(u) = \left(\frac{u}{1+u}\right)^i$, $i = 0,1,2$.*

2. *A*-Statistical approximation

In this section, replacing the limit operator by the A-statistical limit operator and considering a sequence of positive linear operators defined on the space of all real-valued continuous and bounded functions on a subset I^m of $\mathbb{R}^m$, the real m-dimensional space, where $I^m := I \times I \times \cdots \times I$, we give an extension of Theorem A.

To achieve this we first consider the case of $m = 2$.

Let $K := I^2 = [0,\infty) \times [0,\infty)$. Then, the sup norm on $C_B(K)$ is given by

$$\|f\| := \sup_{(x,y)\in K} |f(x,y)|, \quad f \in C_B(K),$$

and also the value of Lf at a point $(x,y) \in K$ is denoted by $L(f(u,v);x,y)$ or simply $L(f;x,y)$.

We consider the modulus of continuity $w_2(f;\delta_1,\delta_2)$ (for the functions of two variables) given by, for any $\delta_1,\ \delta_2 > 0$,

$$\begin{aligned} w_2(f;\delta_1,\delta_2) := \sup\{&|f(u,v)-f(x,y)| : (u,v),(x,y) \in K, \text{ and} \\ &|u-x| \leq \delta_1, |v-x| \leq \delta_2\}. \end{aligned}$$

It is clear that a necessary and sufficient condition for a function $f \in C_B(K)$ is

$$\lim_{\delta_1\to 0,\ \delta_2\to 0} w_2(f;\delta_1,\delta_2) = 0.$$

We now introduce the space H_{w_2} of all real-valued functions f defined on K and satisfying

$$|f(u,v)-f(x,y)| \leq w_2\left(f;\left|\frac{u}{1+u}-\frac{x}{1+x}\right|,\left|\frac{v}{1+v}-\frac{y}{1+y}\right|\right). \tag{2.1}$$

Then observe that any function in H_{w_2} is continuous and bounded on K.

With this terminology we have the following:

Theorem 2.1. *Let $A = (a_{jn})$ be a non-negative regular summability matrix, and let $\{L_n\}$ be a sequence of positive linear operators from H_{w_2} into $C_B(K)$. Then, for any $f \in H_{w_2}$,*

$$st_A - \lim_n \|L_n f - f\| = 0 \tag{2.2}$$

is satisfied if the following holds:

$$st_A - \lim_n \|L_n f_i - f_i\| = 0, \quad i = 0,1,2,3, \tag{2.3}$$

where

$$f_0(u,v) = 1,\ f_1(u,v) = \frac{u}{1+u},\ f_2(u,v) = \frac{v}{1+v},$$
$$f_3(u,v) = \left(\frac{u}{1+u}\right)^2 + \left(\frac{v}{1+v}\right)^2.$$

Proof. Assume that (2.3) holds, and let $f \in H_{w_2}$. By (2.3), for every $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that $|f(u,v) - f(x,y)| < \varepsilon$ holds for all $(u,v) \in K$ satisfying $\left|\frac{u}{1+u} - \frac{x}{1+x}\right| < \delta_1$ and $\left|\frac{v}{1+v} - \frac{y}{1+y}\right| < \delta_2$. Let

$$K_{\delta_1,\delta_2} := \left\{(u,v) \in K\colon \left|\frac{u}{1+u} - \frac{x}{1+x}\right| < \delta_1 \text{ and } \left|\frac{v}{1+v} - \frac{y}{1+y}\right| < \delta_2\right\}.$$

Hence we may write

$$\begin{aligned} |f(u,v) - f(x,y)| &= |f(u,v) - f(x,y)|\,\chi_{K_{\delta_1,\delta_2}}(u,v) \\ &\quad + |f(u,v) - f(x,y)|\,\chi_{K \setminus K_{\delta_1,\delta_2}}(u,v) \\ &< \varepsilon + 2M\chi_{K \setminus K_{\delta_1,\delta_2}}(u,v), \end{aligned} \tag{2.4}$$

where χ_R denotes the characteristic function of the set R and $M := \|f\|$. We also get that

$$\chi_{K \setminus K_{\delta_1,\delta_2}}(u,v) \le \frac{1}{\delta_1^2}\left(\frac{u}{1+u} - \frac{x}{1+x}\right)^2 + \frac{1}{\delta_2^2}\left(\frac{v}{1+v} - \frac{y}{1+y}\right)^2. \tag{2.5}$$

Combining (2.4) with (2.5) we have

$$|f(u,v) - f(x,y)| \le \varepsilon + \frac{2M}{\delta^2}\left\{\left(\frac{u}{1+u} - \frac{x}{1+x}\right)^2 + \left(\frac{v}{1+v} - \frac{y}{1+y}\right)^2\right\}, \tag{2.6}$$

where $\delta := \min\{\delta_1, \delta_2\}$. Using linearity and positivity of the operators L_n we get, for any $n \in \mathbb{N}$,

$$\begin{aligned} |L_n(f;x,y) - f(x,y)| &\le L_n\left(|f(u,v) - f(x,y)|;x,y\right) \\ &\quad + |f(x,y)|\,|L_n(f_0;x,y) - f_0(x,y)|. \end{aligned}$$

Then, by (2.5), we obtain

$$\begin{aligned}
|L_n(f;x,y)-f(x,y)| &\leq \varepsilon L_n(f_0;x,y) \\
&\quad + \frac{2M}{\delta^2}\left\{L_n\left(\left(\frac{u}{1+u}-\frac{x}{1+x}\right)^2;x,y\right)\right. \\
&\quad \left. + L_n\left(\left(\frac{v}{1+v}-\frac{y}{1+y}\right)^2;x,y\right)\right\} \\
&\quad + M\,|L_n(f_0;x,y)-f_0(x,y)|\,.
\end{aligned}$$

By some simple calculations we have

$$\begin{aligned}
|L_n(f;x,y)-f(x,y)| &\leq \varepsilon + (\varepsilon+M)\,|L_n(f_0;x,y)-f_0(x,y)| \\
&\quad + \frac{2M}{\delta^2}\left\{L_n(f_3;x,y)-\frac{2x}{1+x}L_n(f_1;x,y)\right. \\
&\quad - \frac{2y}{1+y}L_n(f_2;x,y) \\
&\quad \left. + \left(\left(\frac{x}{1+x}\right)^2+\left(\frac{y}{1+y}\right)^2\right)L_n(f_0;x,y)\right\} \\
&= \varepsilon + (\varepsilon+M)\,|L_n(f_0;x,y)-f_0(x,y)| \\
&\quad + \frac{2M}{\delta^2}\left(L_n(f_3;x,y)-f_3(x,y)\right) \\
&\quad - \frac{4M}{\delta^2}\left(\frac{x}{1+x}\right)\left(L_n(f_1;x,y)-f_1(x,y)\right) \\
&\quad - \frac{4M}{\delta^2}\left(\frac{y}{1+y}\right)\left(L_n(f_2;x,y)-f_2(x,y)\right) \\
&\quad + \frac{2M}{\delta^2}\left(\left(\frac{x}{1+x}\right)^2+\left(\frac{y}{1+y}\right)^2\right) \\
&\quad \times \left(L_n(f_0;x,y)-f_0(x,y)\right) \\
&\leq \varepsilon + \left(\varepsilon+M+\frac{4M}{\delta^2}\right)|L_n(f_0;x,y)-f_0(x,y)| \\
&\quad + \frac{4M}{\delta^2}\,|L_n(f_1;x,y)-f_1(x,y)| \\
&\quad + \frac{4M}{\delta^2}\,|L_n(f_2;x,y)-f_2(x,y)| \\
&\quad + \frac{2M}{\delta^2}\,|L_n(f_3;x,y)-f_3(x,y)|\,.
\end{aligned}$$

Then, taking supremum over $(x,y)\in K$ we have

$$\begin{aligned}
\|L_n f - f\| &\leq \varepsilon + B\{\|L_n f_0 - f_0\| + \|L_n f_1 - f_1\| \\
&\quad + \|L_n f_2 - f_2\| + \|L_n f_3 - f_3\|\},
\end{aligned} \tag{2.7}$$

where $B := \varepsilon + M + \frac{4M}{\delta^2}$. For a given $r > 0$, choose $\varepsilon > 0$ such that $\varepsilon < r$. Define the following sets:

$$D := \{n\colon \|L_n f - f\| \geq r\},$$

$$D_1 := \left\{n\colon \|L_n f_0 - f_0\| \geq \frac{r-\varepsilon}{4B}\right\},$$

$$D_2 := \left\{n\colon \|L_n f_1 - f_1\| \geq \frac{r-\varepsilon}{4B}\right\},$$

$$D_3 := \left\{n\colon \|L_n f_2 - f_2\| \geq \frac{r-\varepsilon}{4B}\right\},$$

$$D_4 := \left\{n\colon \|L_n f_3 - f_3\| \geq \frac{r-\varepsilon}{4B}\right\}.$$

It follows from (2.7) that $D \subseteq D_1 \cup D_2 \cup D_3 \cup D_4$. Therefore, for each $j \in \mathbb{N}$, we may write

$$\sum_{n\in D} a_{jn} \leq \sum_{n\in D_1} a_{jn} + \sum_{n\in D_2} a_{jn} + \sum_{n\in D_3} a_{jn} + \sum_{n\in D_4} a_{jn}. \tag{2.8}$$

Letting $j \to \infty$ in (2.8) and using (2.3) we conclude that

$$\lim_j \sum_{n\in D} a_{jn} = 0,$$

whence gives (2.2). So the proof is completed. □

Now replace I^2 by $I^m := [0,\infty) \times \cdots \times [0,\infty)$ and consider the modulus of continuity $w_m(f;\delta_1,\ldots,\delta_m)$ (for the functions f of m-variables) given by, for any $\delta_1,\ldots,\delta_m > 0$,

$$\begin{aligned} w_m(f;\delta_1,\ldots,\delta_m) := \sup\{&|f(u_1,\ldots,u_m) - f(x_1,\ldots,x_m)|\colon \\ &(u_1,\ldots,u_m),(x_1,\ldots,x_m) \in I^m \\ &\text{and } |u_i - x_i| \leq \delta_i,\ i = 1,2,\ldots,m\}. \end{aligned}$$

Then let H_{w_m} be the space of all real-valued functions f satisfying

$$\begin{aligned} &|f(u_1,\ldots,u_m) - f(x_1,\ldots,x_m)| \\ &\quad \leq w_2\left(f;\left|\frac{u_1}{1+u_1} - \frac{x_1}{1+x_1}\right|,\ldots,\left|\frac{u_m}{1+u_m} - \frac{x_m}{1+x_m}\right|\right). \end{aligned}$$

Therefore, using a similar technique in the proof of Theorem 2.1 one can obtain the following result immediately.

Theorem 2.2. *Let $A = (a_{jn})$ be a non-negative regular summability matrix, and let $\{L_n\}$ be a sequence of positive linear operators from H_{w_m} into $C_B(I^m)$. Then, for any function $f \in H_{w_m}$,*

$$st_A - \lim_n \|L_n f - f\| = 0$$

is satisfied if the following holds:

$$st_A - \lim_n \|L_n f_i - f_i\| = 0, \quad i = 0,1,\ldots,m+1,$$

where

$$f_0(u_1,\dots,u_m)=1,\, f_i(u_1,\dots,u_m)=\frac{u_i}{1+u_i},\quad i=1,2,\dots,m,$$
$$f_{m+1}(u_1,\dots,u_m)=\sum_{k=1}^{m}\left(\frac{u_k}{1+u_k}\right)^2.$$

Remark. If we choose $m=1$, $I=[0,\infty)$ and replace $A=(a_{jn})$ by the identity matrix, then Theorem 2.2 reduces to Theorem A.

3. Concluding remarks

In this section we display an example such that Theorem A does not work but Theorem 2.1 does.

Let $A=(a_{jn})$ be a non-negative regular summability matrix such that $\lim_j \max_k\{a_{jn}\}=0$. In this case we know from [13] that A-statistical convergence is stronger than ordinary convergence. So, we can choose a non-negative sequence (u_n) which converges A-statistically to 1 but non-convergent. Assume that $I=[0,\infty)$ and $K:=I^2=I\times I$. We now consider the following positive linear operators defined on $H_{w_2}(K)$:

$$T_n(f;x,y)=\frac{u_n}{(1+x)^n(1+y)^n}\sum_{k=0}^{n}\sum_{l=0}^{n} f\left(\frac{k}{n-k+1},\frac{l}{n-l+1}\right)\binom{n}{k}\binom{n}{l}x^k y^l,$$

where $f\in H_{w_2}$, $(x,y)\in K$ and $n\in\mathbb{N}$.

We should remark that in the case of $u_n=1$, the operators T_n turn out to be the operators of Bleimann, Butzer and Hahn [2] (of two variables).

Since

$$(1+x)^n=\sum_{k=0}^{n}\binom{n}{k}x^k,$$

it is clear that, for all $n\in\mathbb{N}$,

$$T_n(f_0;x,y)=u_n.$$

Now, by assumption we have

$$st_A-\lim_n\|T_n f_0-f_0\|=st_A-\lim_n|u_n-1|=0. \tag{3.1}$$

Using the definition of T_n, we get

$$\begin{aligned}T_n(f_1;x,y)&=\frac{u_n}{(1+x)^n(1+y)^n}\sum_{k=1}^{n}\frac{k}{n+1}\binom{n}{k}x^k\sum_{l=0}^{n}\binom{n}{l}y^l\\&=\frac{u_n x}{(1+x)^n}\sum_{k=0}^{n-1}\frac{k+1}{n+1}\binom{n}{k+1}x^k.\end{aligned}$$

Since

$$\binom{n}{k+1}=\frac{n}{k+1}\binom{n-1}{k},$$

we obtain

$$T_n(f_1;x,y) = \frac{u_n x}{1+x}\frac{1}{(1+x)^{n-1}}\sum_{k=0}^{n-1}\frac{n}{n+1}\binom{n-1}{k}x^k$$
$$= \frac{nu_n}{n+1}\left(\frac{x}{1+x}\right),$$

which yields

$$|T_n(f_1;x,y) - f_1(x,y)| = \frac{x}{1+x}\left|\frac{n}{n+1}u_n - 1\right|$$

and hence

$$\|T_n f_1 - f_1\| \le \left|\frac{n}{n+1}u_n - 1\right|. \tag{3.2}$$

Since $\lim_n \frac{n}{n+1} = 1$ and $st_A - \lim_n u_n = 1$, observe that $st_A - \lim_n \frac{n}{n+1}u_n = 1$, so it follows from (3.2) that

$$st_A - \lim_n \|T_n f_1 - f_1\| = 0. \tag{3.3}$$

Similarly, we get

$$st_A - \lim_n \|T_n f_2 - f_2\| = 0. \tag{3.4}$$

Finally, we claim that

$$st_A - \lim_n \|T_n f_3 - f_3\| = 0. \tag{3.5}$$

Indeed, by the definition of T_n we have

$$T_n(f_3;x,y) = \frac{u_n}{(1+x)^n(1+y)^n}\sum_{k=0}^{n}\sum_{l=0}^{n}\left[\frac{k^2}{(n+1)^2} + \frac{l^2}{(n+1)^2}\right]\binom{n}{k}\binom{n}{l}x^k y^l$$
$$= \frac{u_n}{(1+x)^n(1+y)^n}\sum_{k=1}^{n}\frac{k^2}{(n+1)^2}\binom{n}{k}x^k\sum_{l=0}^{n}\binom{n}{l}y^l$$
$$+ \frac{u_n}{(1+x)^n(1+y)^n}\sum_{k=0}^{n}\binom{n}{k}x^k\sum_{l=1}^{\infty}\frac{l^2}{(n+1)^2}\binom{n}{l}y^l$$
$$= \frac{u_n}{(1+x)^n}\sum_{k=2}^{n}\frac{k(k-1)}{(n+1)^2}\binom{n}{k}x^k + \frac{u_n}{(1+x)^n}\sum_{k=1}^{n}\frac{k}{(n+1)^2}x^k$$
$$+ \frac{u_n}{(1+y)^n}\sum_{l=2}^{n}\frac{l(l-1)}{(n+1)^2}\binom{n}{l}y^l + \frac{u_n}{(1+y)^n}\sum_{l=1}^{n}\frac{l}{(n+1)^2}y^l$$

$$= \frac{u_n x^2}{(1+x)^n}\sum_{k=2}^{n-2}\frac{(k+2)(k+1)}{(n+1)^2}\binom{n}{k+2}x^k + \frac{u_n x}{(1+x)^n}\sum_{k=0}^{n-1}\frac{k+1}{(n+1)^2}x^k$$
$$+ \frac{u_n y^2}{(1+y)^n}\sum_{l=0}^{n-2}\frac{(l+2)(l+1)}{(n+1)^2}\binom{n}{l+2}y^l + \frac{u_n y}{(1+y)^n}\sum_{l=1}^{n-1}\frac{l+1}{(n+1)^2}y^l.$$

Now using the facts that

$$\binom{n}{k+2}=\frac{n(n-1)}{(k+1)(k+2)}\binom{n-2}{k},\ \binom{n}{k+1}=\frac{n}{k+1}\binom{n-1}{k},$$
$$\binom{n}{l+2}=\frac{n(n-1)}{(l+1)(l+2)}\binom{n-2}{l},\ \binom{n}{l+1}=\frac{n}{l+1}\binom{n-1}{l},$$

we get

$$\begin{aligned}T_n(f_3;x,y)&=\frac{n(n-1)u_n}{(n+1)^2}\frac{x^2}{(1+x)^2}\frac{1}{(1+x)^{n-2}}\sum_{k=0}^{n-2}\binom{n-2}{k}x^k\\&\quad+\frac{nu_n}{(n+1)^2}\frac{x}{1+x}\frac{1}{(1+x)^{n-1}}\sum_{k=0}^{n-1}\binom{n-1}{k}x^k\\&\quad\times\frac{n(n-1)u_n}{(n+1)^2}\frac{y^2}{(1+y)^2}\frac{1}{(1+y)^{n-2}}\sum_{k=0}^{n-2}\binom{n-2}{l}y^l\\&\quad+\frac{nu_n}{(n+1)^2}\frac{y}{1+y}\frac{1}{(1+y)^{n-1}}\sum_{k=0}^{n-1}\binom{n-1}{l}y^l\\&=\frac{n(n-1)u_n}{(n+1)^2}\frac{x^2}{(1+x)^2}+\frac{nu_n}{(n+1)^2}\frac{x}{1+x}\\&\quad+\frac{n(n-1)u_n}{(n+1)^2}\frac{y^2}{(1+y)^2}+\frac{nu_n}{(n+1)^2}\frac{y}{1+y},\end{aligned}$$

which implies that

$$\begin{aligned}|T_n(f_3;x,y)-f_3(x,y)|&=\left(\frac{x^2}{(1+x)^2}+\frac{y^2}{(1+y)^2}\right)\left|\frac{n(n-1)u_n}{(n+1)^2}-1\right|\\&\quad+\left(\frac{x}{1+x}+\frac{y}{1+y}\right)\frac{nu_n}{(n+1)^2}.\end{aligned}$$

Taking supremum over $(x,y)\in K$ we have

$$\|T_nf_3-f_3\|\le 2(\alpha_n+\beta_n), \tag{3.6}$$

where $\alpha_n:=\left|\frac{n(n-1)u_n}{(n+1)^2}-1\right|$ and $\beta_n:=\frac{nu_n}{(n+1)^2}$. Then, by assumption, observe that

$$st_A-\lim_n\alpha_n=st_A-\lim_n\beta_n=0. \tag{3.7}$$

Now given $\varepsilon>0$, define the following sets

$$U:=\{n\colon \|T_nf_3-f_3\|\ge\varepsilon\},$$
$$U_1:=\left\{n\colon \alpha_n\ge\frac{\varepsilon}{4}\right\},\quad U_2:=\left\{n\colon \beta_n\ge\frac{\varepsilon}{4}\right\}.$$

By (3.6), it is obvious that $U\subseteq U_1\cup U_2$. Then, for each $j\in\mathbb{N}$, we may write that

$$\sum_{n\in U}a_{jn}\le\sum_{n\in U_1}a_{jn}+\sum_{n\in U_2}a_{jn}. \tag{3.8}$$

Taking limit as $j \to \infty$ in (3.8) and using (3.7) we have

$$\lim_j \sum_{n \in U} a_{jn} = 0,$$

which proves (3.5).

Therefore, using (3.1), (3.3), (3.4) and (3.5) in Theorem 2.1, we obtain that, for all $f \in H_{w_2}$,

$$st_A - \lim_n \|T_n f - f\| = 0.$$

However, since the sequence (u_n) is non-convergent, $\{T_n f\}$ is not uniformly convergent to f.

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